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# Dirac operators in Boson–Fermion Fock spaces and supersymmetric quantum field theory

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Infinite dimensional analysis is developed on an abstract Boson–Fermion Fock space. A general class of Dirac operators acting there is introduced and properties of them are investigated. An index theorem for the Dirac operators is established in terms of a path integral on a loop space. It is shown that the abstract formalism presented here gives a mathematical unification for some models of supersymmetric quantum field theory.

*Keywords:* *infinite dimensional analysis, Dirac operator, Boson–Fermion Fock space, path (functional) integral, index theorem, supersymmetric quantum field theory*
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*Dedicated to Professor Hiroshi Ezawa on the occasion of his 60th birthday*

## 1. Introduction

In ref. [1], the author initiated a new analysis on Boson–Fermion Fock space (BFFS), introducing a class of infinite dimensional Dirac operators acting there. The analysis was further developed in ref. [2] to define a more general class of Dirac operators acting in a most general BFFS. One of the purposes of these studies was to establish index theorems in a framework of infinite dimensional analysis. The topics discussed in refs. [1,2] include: (1) trace formulae with respect to "Gibbs states" in both the Boson and Fermion Fock spaces; (2) operator-theoretical analysis of de Rham and Dirac operators, and Laplacians acting in BFFS; (3) index theorems for the Dirac operators in terms of path integral representations. The theory and the methods presented in ref. [2] have been applied to various directions [3–7] and are still in progress [15–17].

The physical background of the work in refs. [1,2] is in supersymmetric quantum field theory (SSQFT) (e.g., [40–42]). From the point of view of constructive QFT, it is interesting to show the mathematical existence of SSQFT models.

For the Wess–Zumino (WZ) models on a cylinder, this has been done by Jaffe et al. [26–30]. We have shown that the abstract formalism in ref. [2] clarifies the mathematical structures of some SSQFT models including the WZ models, giving a mathematical unification for them (see section 9 in the present paper).

The theory in ref. [2] has a generalization to the case where the Boson Fock space part is replaced by the $L^2$-space on a probability space with a non-Gaussian measure (a Boson Fock space corresponds to a Gaussian measure) and the resulting theory can be applied to the problem of supersymmetric extension of scalar QFT [3,15].

In this paper we summarize some of the results obtained in refs. [1,2,4,5,17]. In section 2, we review some basic facts in Fock space theory and describe trace formulae for Gibbs states. Section 3 is concerned with a sequence of de Rham type operators and their associated Laplacians in a sequence of Hilbert spaces of which the infinite direct sum is a general BFFS. The sequence of the de Rham operators can be lifted to the BFFS to define a de Rham type operator acting there, which, in turn, is used to define a "free" Dirac operator in the BFFS. These will be done in section 4. We also introduce the Laplacian associated with the de Rham operator. In section 5 we take an interlude to present some basic facts in the theory of Fredholm operators and in supersymmetric quantum theory, which should clarify a connection of our infinite dimensional analysis with index theory and supersymmetry. In section 6 we state a result on the Fredholm property of the free Dirac operator introduced in section 4. Section 7 is devoted to the description of a perturbation of the free Dirac operator. We give an explicit form of the Laplacian associated with the perturbed Dirac operator. In section 8 we present a result on the path integral representation of the index of the perturbed Dirac operator. In the last section we discuss some examples in SSQFT to illustrate how our abstract formalism unifies mathematically some SSQFT models.

## 2. Fock spaces and trace formulae

### 2.1. BOSON FOCK SPACE

We use the $Q$-space description of Boson Fock space [36, ch. I]. Let $\mathcal{H}$ be a real separable Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and $\{\phi(f)|f \in \mathcal{H}\}$ be the Gaussian mean zero random process indexed by $\mathcal{H}$, i.e., $\{\phi(f)|f \in \mathcal{H}\}$ is a family of random variables on a probability measure space $(E, \mathbf{B}(E), \mu)$ such that the mapping $f \to \phi(f)$ is linear, the Borel field $\mathbf{B}(E)$ is generated by $\{\phi(f)|f \in \mathcal{H}\}$, and

$$\int_E e^{i\phi(f)}\, d\mu = \exp(-\|f\|_{\mathcal{H}}^2/2), \quad f \in \mathcal{H}. \tag{2.1}$$

We denote by $\mathcal{H}_c$ the complexification of $\mathcal{H}$ and by $(\cdot,\cdot)_{\mathcal{H}_c}$ its inner product (complex linear in the second variable). Each element $f$ of $\mathcal{H}_c$ is uniquely written as $f = f_1 + \mathrm{i}f_2$ with $f_1, f_2 \in \mathcal{H}$ ($\mathrm{i} = \sqrt{-1}$). Then we define $\phi(f)$ by $\phi(f) = \phi(f_1) + \mathrm{i}\phi(f_2)$.

Let $\Gamma_0(\mathcal{H}) = \mathbb{C}$ and

$$\Gamma_n(\mathcal{H}) = \overline{\mathcal{L}\{:\phi(f_1)\cdots\phi(f_n): \mid f_j \in \mathcal{H}, j = 1,\ldots,n\}}, \quad n \geq 1, \tag{2.2}$$

where $\mathcal{L}\{\cdots\}$ denotes the linear span of elements in $\{\cdots\}$, $\overline{\mathcal{L}\{\cdots\}}$ the closure of $\mathcal{L}\{\cdots\}$, and $:\phi(f_1)\cdots\phi(f_n):$ the "Wick product" with respect to (w.r.t.) the measure $\mu$ of the random variables $\phi(f_1),\ldots,\phi(f_n)$ [36]. The Boson Fock space $L^2(E,\mathrm{d}\mu)$ is decomposed as

$$L^2(E,\mathrm{d}\mu) = \bigoplus_{n=0}^{\infty} \Gamma_n(\mathcal{H}). \tag{2.3}$$

For a self-adjoint operator $A$ in $\mathcal{H}_c$ with domain $D(A)$, the second quantization $\mathrm{d}\Gamma_{\mathrm{b}}(A)$ in $L^2(E,\mathrm{d}\mu)$ is defined as the self-adjoint operator which is reduced by each $\Gamma_n(\mathcal{H})$ with the reduced part $\mathrm{d}\Gamma_{\mathrm{b}}^{(n)}(A)$ being of the form $\mathrm{d}\Gamma_{\mathrm{b}}^{(0)}(A)1 = 0$,

$$\mathrm{d}\Gamma_{\mathrm{b}}^{(n)}(A):\phi(f_1)\cdots\phi(f_n): = \sum_{j=1}^{n} :\phi(f_1)\cdots\phi(Af_j)\cdots\phi(f_n):, \quad n \geq 1,$$
$$f_j \in D(A), j = 1,\ldots,n. \tag{2.4}$$

We denote by $\mathbb{P}_n$ the set of all complex polynomials of $n$ variables. For an operator $T$ from $\mathcal{H}_c$ to another space, we define a subspace $\mathcal{P}_T$ of $L^2(E,\mathrm{d}\mu)$ by

$$\mathcal{P}_T = \mathcal{L}\{P(\phi(f_1),\ldots,\phi(f_n)) \mid P \in \mathbb{P}_n, f_j \in D(T), j = 1,\ldots,n, n \geq 0\}. \tag{2.5}$$

It follows that, if $T$ is densely defined, then $\mathcal{P}_T$ is dense in $L^2(E,\mathrm{d}\mu)$. For $T = I$ (identity), we set

$$\mathcal{P}_I = \mathcal{P}. \tag{2.6}$$

For a Hilbert space $\mathcal{M}$, we denote by $L^2(E,\mathrm{d}\mu;\mathcal{M})$ the Hilbert space of $\mathcal{M}$-valued square integrable functions on $(E,\mathrm{d}\mu)$.

We define an operator $\nabla$ from $L^2(E,\mathrm{d}\mu)$ to $L^2(E,\mathrm{d}\mu;\mathcal{H}_c)$ with domain $\mathcal{P}$ by

$$\nabla P(\phi(f_1),\ldots,\phi(f_n)) = \sum_{j=1}^{n} (\partial_j P)(\phi(f_1),\ldots,\phi(f_n)) f_j \tag{2.7}$$

for $P \in \mathbb{P}_n$ and $f_j \in \mathcal{H}_c, j = 1,\ldots,n$, and extending it by linearity to $\mathcal{P}$, where $\partial_j P$ denotes the partial derivative of $P = P(z_1,\ldots,z_n)$ w.r.t. $z_j \in \mathbb{C}$ $(j = 1,\ldots,n)$. The well-definedness of $\nabla$ (i.e., if $\Psi$ and $\Phi$ in $\mathcal{P}$ satisfy $\Psi = \Phi$ a.e. (almost everywhere), then $\nabla\Psi = \nabla\Phi$ a.e.) is ensured by the following integration by parts formula w.r.t. the measure $\mu$:

$$\int_E (f, \nabla\Psi)_{\mathcal{H}_c}\Phi \, d\mu = -\int_E \Psi(f, \nabla\Phi)_{\mathcal{H}_c} \, d\mu + \int_E \phi(J_{\mathcal{H}}f)\Psi\Phi \, d\mu, \quad \Psi, \Phi \in \mathcal{P}, f \in \mathcal{H}_c, \tag{2.8}$$

where $J_{\mathcal{H}}$ denotes the natural conjugation on $\mathcal{H}_c$: $J_{\mathcal{H}}(f_1 + \mathrm{i}f_2) := f_1 - \mathrm{i}f_2, f_1, f_2 \in \mathcal{H}$. Formula (2.8) can be proven in the same way as in the proof of ref. [21, th. 6.3.1].

For each $f \in \mathcal{H}_c$, we define an operator $\tilde{\nabla}_f$ in $L^2(E, d\mu)$ with domain $\mathcal{P}$ by

$$\tilde{\nabla}_f \Psi = (f, \nabla\Psi)_{\mathcal{H}_c}. \tag{2.9}$$

Note that $\tilde{\nabla}_f$ is *complex anti-linear* in $f$.

In general, we denote by $\mathcal{V} \hat{\otimes} \mathcal{W}$ the algebraic tensor product of vector spaces $\mathcal{V}$ and $\mathcal{W}$, and by $T^*$ the adjoint of the densely defined linear operator $T$.

**Lemma 2.1.** *For all $f \in \mathcal{H}_c$, $\tilde{\nabla}_f$ and $\nabla$ are closable with*

$$D(\tilde{\nabla}_f^*) \supset \mathcal{P}, \quad D(\nabla^*) \supset \mathcal{P} \hat{\otimes} \mathcal{H}_c, \tag{2.10}$$

$$\tilde{\nabla}_f^* \Psi = \nabla^*(\Psi \otimes f) = -\tilde{\nabla}_{J_{\mathcal{H}}f}\Psi + \phi(f)\Psi, \quad \Psi \in \mathcal{P}, f \in \mathcal{H}_c. \tag{2.11}$$

*Proof.* Using (2.8), one first proves (2.10) and (2.11). Relation (2.10) implies that $D(\tilde{\nabla}_f^*)$ and $D(\nabla^*)$ are dense in $L^2(E, d\mu)$ and $L^2(E, d\mu; \mathcal{H}_c)$, respectively. Hence, by a general criterion on the closability of linear operators (e.g., [32, ch. III, §5, th. 5.28], [34, th. VIII.1]), $\tilde{\nabla}_f$ and $\nabla$ are closable. □

We denote the closure of $\tilde{\nabla}_f$ and of $\nabla$ by the same symbol, respectively.

We next describe trace formulae concerning the "heat operator" $\exp(-\beta H)$ ($\beta > 0$) for a class of self-adjoint operators $H$ in $L^2(E, d\mu)$. We shall denote by $\mathcal{I}_p(\mathcal{M})$ the $p$th Schatten class on the Hilbert space $\mathcal{M}$ [i.e., $T \in \mathcal{I}_p(\mathcal{M}) \iff (T^*T)^{p/2}$ is trace class on $\mathcal{M}$] and by $\|\cdot\|_p$ the norm:

$$\|T\|_p = \{\mathrm{Tr}(T^*T)^{p/2}\}^{1/p}, \quad T \in \mathcal{I}_p(\mathcal{M}), \tag{2.12}$$

where Tr denotes trace. For $T \in \mathcal{I}_1(\mathcal{M})$ [$\mathcal{I}_p(\mathcal{M}), p \geq 2$], $\det(I + T)$ [$\det_p(1 + T)$] denotes the [regularized] determinant of $I + T$ (e.g., [37, 38]).

*In the rest of this subsection, we assume that $A$ is a strictly positive self-adjoint operator in $\mathcal{H}$ such that $A^{-\gamma} \in \mathcal{I}_1(\mathcal{H})$ for some $\gamma > 0$.* Then one can easily show that, for all $\beta > 0$, $\exp(-\beta A) \in \mathcal{I}_1(\mathcal{H}_c)$ and $\exp(-\beta \, d\Gamma_b(A)) \in \mathcal{I}_1(L^2(E, d\mu))$ with

$$Z_A(\beta) := \mathrm{Tr}\, e^{-\beta \, d\Gamma_b(A)} = \frac{1}{\det(I - e^{-\beta A})}. \tag{2.13}$$

Let $\delta > \gamma$ and $\mathcal{H}_{-\delta}(A)$ be the completion of $\mathcal{H}$ in the norm $\|\cdot\|_{-\delta} = \|A^{-\delta/2} \cdot\|_{\mathcal{H}}$. Then the dual space of $\mathcal{H}_{-\delta}(A)$ can be identified with $\mathcal{H}_\delta(A) :=$

$D(A^{\delta/2})$ equipped with the inner product $(\cdot,\cdot)_\delta = (A^{\delta/2}\cdot, A^{\delta/2}\cdot)_{\mathcal{H}}$. The embedding $\mathcal{H} \to \mathcal{H}_{-\delta}(A)$ is nuclear. Hence, by a theorem of Sazonov, Minlos and Gross [23], we can take the measure space $E$ to be

$$E = \mathcal{H}_{-\delta}(A). \tag{2.14}$$

Moreover, for all $f \in \mathcal{H}_\delta(A)$, the random variable $\phi(f)$ is realized as $\phi(f) = \langle\phi, f\rangle$, the duality pairing between $\phi \in \mathcal{H}_{-\delta}(A)$ and $f$, which coincides with the inner product $(\phi, f)_{\mathcal{H}}$ if $\phi \in \mathcal{H}$.

Let $\beta > 0$ and

$$E_\beta = C([0,\beta];E) \tag{2.15}$$

be the space of $E$-valued continuous functions on $[0,\beta]$. In the same way as in the proof of ref. [24, prop. 5.1], we can prove the following fact:

**Proposition 2.2** [1,2]. *There exists a Gaussian mean zero process $\Phi_t$ on $[0,\beta]$ with state space $E$ and with continuous sample paths such that for all $f, g \in \mathcal{H}_\delta(A)$,*

$$\begin{aligned}\int_{E_\beta} &\mathrm{d}\mu_\beta(\Phi)\langle\Phi_t, f\rangle\langle\Phi_s, g\rangle \\ &= (f, (1-\mathrm{e}^{-\beta A})^{-1}(\mathrm{e}^{-|t-s|A} + \mathrm{e}^{-(\beta-|t-s|)A})g)_{\mathcal{H}}, \\ &\quad t,s \in [0,\beta],\end{aligned} \tag{2.16}$$

*where $\mathrm{d}\mu_\beta$ denotes the underlying measure on $E_\beta$.*

**Remark 2.3.** We can show that $\Phi_0 = \Phi_\beta$ for a.e. $\Phi$, which implies that, for a.e. $\Phi$, the mapping $t \to \Phi_t$ is a loop in $E$. Hence the measure $\mu_\beta$ can be regarded as a measure on the loop space of $E$.

**Definition 2.4.** Let $H$ be a self-adjoint operator in $L^2(E,\mathrm{d}\mu)$ such that $\mathrm{e}^{-tH}$ is in $\mathcal{I}_1(L^2(E,\mathrm{d}\mu))$ for all $t > 0$. We say that a measurable function $G$ on $E$ is in the set $\mathcal{I}_H$ if, for all $\varepsilon > 0$, $\exp(-\varepsilon H)|G|\exp(-\varepsilon H)$ defines a unique trace class operator on $L^2(E,\mathrm{d}\mu)$.

Let $V$ be a real-valued measurable function on $E$. Suppose that there exists a dense subspace $D$ in $L^2(E,\mathrm{d}\mu)$ such that $D \subset D(\mathrm{d}\Gamma_\mathrm{b}(A)) \cap D(V)$ and

$$H_V = \mathrm{d}\Gamma_\mathrm{b}(A) + V \tag{2.17}$$

is bounded from below on $D$. We denote the Friedrichs extension (e.g., [35, §X.3]) of $H_V \restriction D$ by the same symbol $H_V$. A basic trace formula is given in the following theorem.

**Theorem 2.5** [1,2,17]. *Let $\beta > 0$ and $0 < t_1 < t_2 < \cdots < t_n < \beta$. Let $V$ be as above and suppose that, for all $t \in (0, \beta]$,*

$$\int_{E_t} \mathrm{d}\mu_t(\Phi) \exp\left( - \int_0^t V(\Phi_s)\,\mathrm{d}s \right) < \infty.$$

*Then, for all $t > 0$, $\mathrm{e}^{-tH_V} \in \mathcal{I}_1(L^2(E, \mathrm{d}\mu))$ and, for all $G_1, \ldots, G_n \in \mathcal{I}_{H_V}$,*

$$\frac{\mathrm{Tr}\,(\mathrm{e}^{-t_1 H_V} G_1 \mathrm{e}^{-(t_2-t_1)H_V} G_2 \cdots G_n \mathrm{e}^{-(\beta - t_n)H_V})}{Z_A(\beta)} = \int_{E_\beta} \mathrm{d}\mu_\beta(\Phi)\, G_1(\Phi_{t_1}) \cdots \tilde{G}_n(\Phi_{t_n}) \exp\left( - \int_0^\beta V(\Phi_t)\,\mathrm{d}t \right). \tag{2.18}$$

This theorem is a refinement of ref. [2, prop. D.3] in the sense that the conditions which imply (2.18) are weakened. For the details of the proof, see ref. [17].

### 2.2. FERMION FOCK SPACE

Let $\mathcal{K}$ be a real separable Hilbert space and $\wedge^p(\mathcal{K}_c)$ be the $p$-fold anti-symmetric tensor product of $\mathcal{K}_c$ [$\wedge^0(\mathcal{K}_c) \equiv \mathbb{C}$]. The Fermion Fock space $\wedge(\mathcal{K}_c)$ over $\mathcal{K}_c$ is defined by

$$\wedge(\mathcal{K}_c) = \bigoplus_{p=0}^{\infty} \wedge^p(\mathcal{K}_c) \tag{2.19}$$

(e.g., [34, §II.4]). For $u_j \in \mathcal{K}_c$, $j = 1, \ldots, p$, we define their exterior product $u_1 \wedge \cdots \wedge u_p \in \wedge^p(\mathcal{K}_c)$ by

$$u_1 \wedge \cdots \wedge u_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)} = A_p(u_1 \otimes u_2 \cdots \otimes u_p), \tag{2.20}$$

where $\mathfrak{S}_p$ is the symmetric group of order $p$, $\varepsilon(\sigma)$ the sign of the permutation $\sigma \in \mathfrak{S}_p$, and $A_p$ is the antisymmetrization operator: $A_p = \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma)\sigma/p!$.

We shall denote by $b(u)$ ($u \in \mathcal{K}_c$) the Fermion annihilation operator on $\wedge(\mathcal{K}_c)$, i.e., $b(u)$ is the bounded linear operator on $\wedge(\mathcal{K}_c)$ such that its adjoint $b(u)^*$, the Fermion creation operator, maps $\wedge^p(\mathcal{K}_c)$ to $\wedge^{p+1}(\mathcal{K}_c)$ for all $p \geq 0$ with the action

$$b(u)^* u_1 \wedge u_2 \wedge \cdots \wedge u_p = \sqrt{p+1}\, u \wedge u_1 \wedge \cdots \wedge u_p, \quad u_j \in \mathcal{K}_c, \quad j = 1, \ldots, p. \tag{2.21}$$

The following anti-commutation relations hold:

$$\{b(u), b(v)^*\} = (u, v)_{\mathcal{K}_c}, \tag{2.22}$$

$$\{b(u), b(v)\} = 0 = \{b(u)^*, b(v)^*\}, \quad u, v \in \mathcal{K}_c, \tag{2.23}$$

where $\{A, B\} = AB + BA$. We denote by $b^{\#}(u)$ either $b(u)$ or $b(u)^*$. It follows from (2.22) that the operator norm $\|b(u)^{\#}\|$ of $b(u)^{\#}$ is given by

$$\|b(u)^{\#}\| = \|u\|_{\mathcal{K}_c}, \quad u \in \mathcal{K}_c. \tag{2.24}$$

For a self-adjoint operator $B$ in $\mathcal{K}_c$, the second quantization $d\Gamma_f(B)$ of $B$ in the Fermion Fock space $\wedge(\mathcal{K}_c)$ is defined as the self-adjoint operator in $\wedge(\mathcal{K}_c)$ which is reduced by each $\wedge^p(\mathcal{K}_c)$ with the reduced part $d\Gamma_f^{(p)}(B)$ being of the form (e.g., [34, §VIII.10]) $d\Gamma_f^{(0)}(B) = 0$,

$$d\Gamma_f^{(p)}(B) = \sum_{j=1}^{p} I \otimes \cdots \otimes I \otimes \overset{j}{\breve{B}} \otimes I \otimes \cdots \otimes I, \quad p \geq 1. \tag{2.25}$$

The number operator on $\wedge(\mathcal{K}_c)$ is defined by

$$N_f = d\Gamma_f(I). \tag{2.26}$$

We next define quadratic operators in $\wedge(\mathcal{K}_c)$. For a densely defined linear operator $T$ in $\mathcal{K}_c$, we define

$$\begin{aligned}\wedge_f(\mathcal{K}_c; T) = \{\Psi = \{\Psi^{(p)}\}_{p=0}^{\infty} | \Psi^{(p)} \in A_p(D(T) \hat{\otimes} \cdots \hat{\otimes} D(T)),\\ \Psi^{(p)} = 0 \text{ for all but finitely many } p\},\end{aligned} \tag{2.27}$$

which is dense in $\wedge(\mathcal{K}_c)$. We set $\wedge_f(\mathcal{K}_c; I) = \wedge_f(\mathcal{K}_c)$. Let $K \in \mathcal{I}_2(\mathcal{K}_c)$. Then there exist orthonormal systems $\{u_n\}_{n=1}^N$ and $\{v_n\}_{n=1}^N$ in $\mathcal{K}_c$ and positive numbers $\{\lambda_n\}_{n=1}^N$ ($N$ may be infinite) such that $\sum_{n=1}^N \lambda_n^2 < \infty$ and

$$K = \sum_{n=1}^{N} \lambda_n (u_n, \cdot\,)_{\mathcal{K}_c} v_n, \tag{2.28}$$

where, in the case $N = \infty$, the r.h.s. converges in the norm of $\mathcal{I}_2(\mathcal{K}_c)$ (cf. [34, th. VI.17]). We define

$$\begin{aligned}\langle b^*|K|b\rangle &= \sum_{n=1}^{N} \lambda_n b(v_n)^* b(u_n),\\ \langle b|K|b\rangle &= \sum_{n=1}^{N} \lambda_n b(J_{\mathcal{K}} v_n) b(u_n),\\ \langle b^*|K|b^*\rangle &= \sum_{n=1}^{N} \lambda_n b(v_n)^* b(J_{\mathcal{K}} u_n)^*,\end{aligned} \tag{2.29}$$

where, in the case $N = \infty$, we can show that the r.h.s.'s strongly converge on $\wedge_f(\mathcal{K}_c)$ and the limits are independent of the choice of representation (2.28) of $K$. For notational simplicity, we denote by $\langle b^{\#}|K|b^{\#}\rangle$ any of these three operators. It is easy to see that $\wedge_f(\mathcal{K}_c) \subset D(\langle b^*|K|b\rangle^*) \cap D(\langle b^*|K|b^*\rangle^*) \cap D(\langle b|K|b\rangle^*)$ and

$$\langle b^{\#}|K|b^{\natural}\rangle^* = \langle b^{\natural *}|K^*|b^{\# *}\rangle \quad \text{on } \wedge_f(\mathcal{K}_c). \tag{2.30}$$

Hence $\langle b^{\#}|K|b^{\#}\rangle$ is closable. We denote its closure by the same symbol. An estimate for $\langle b^{\#}|K|b^{\#}\rangle$ is given in the following lemma:

**Lemma 2.6** [17]. *Let $K \in \mathcal{I}_2(\mathcal{K}_c)$. Then $D(N_f^{1/2}) \subset D(\langle b^{\#}|K|b^{\#}\rangle)$ and*

$$\|\langle b^{\#}|K|b^{\#}\rangle\Psi\| \le \|K\|_2\|(N_f + 2)^{1/2}\Psi\|, \quad \Psi \in D(N_f^{1/2}). \tag{2.31}$$

We introduce a class of Hilbert–Schmidt operators on $\mathcal{K}_c$:

**Lemma 2.7**. *For every $\psi \in \mathcal{K}_c \otimes \mathcal{K}_c$, there exists a unique Hilbert–Schmidt operator $C_\psi$ on $\mathcal{K}_c$ such that, for all $u, v \in \mathcal{K}_c$,*

$$(v, C_\psi u)_{\mathcal{K}_c} = (v \otimes J_{\mathcal{K}} u, \psi)_{\mathcal{K}_c \otimes \mathcal{K}_c}. \tag{2.32}$$

*Moreover,*

$$\|C_\psi\|_2 = \|\psi\|_{\mathcal{K}_c \otimes \mathcal{K}_c}. \tag{2.33}$$

*Proof.* The existence of $C_\psi$ can be proven by employing the Riesz lemma. To prove (2.33) is an easy exercise. □

### 2.3. BFFS

The BFFS we are concerned with is given by

$$\bigwedge(E, \mathcal{K}) = L^2(E, d\mu) \otimes \bigwedge(\mathcal{K}_c) = L^2(E, d\mu; \bigwedge(\mathcal{K}_c)). \tag{2.34}$$

By (2.19) we have the orthogonal decomposition

$$\bigwedge(E, \mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^p(E, \mathcal{K}) \tag{2.35}$$

with

$$\bigwedge^p(E, \mathcal{K}) = L^2(E, d\mu) \otimes \bigwedge^p(\mathcal{K}_c) = L^2(E, d\mu; \bigwedge^p(\mathcal{K}_c)). \tag{2.36}$$

Geometrically, the Hilbert space $\bigwedge(E, \mathcal{K})$ $[\bigwedge^p(E, \mathcal{K})]$ can be regarded as the space of square integrable cross-sections w.r.t. the measure $\mu$ of the product vector bundle $E \times \bigwedge(\mathcal{K}_c)$ $[E \times \bigwedge^p(\mathcal{K}_c)]$.

We can decompose $\bigwedge(E, \mathcal{K})$ as

$$\bigwedge(E, \mathcal{K}) = \bigwedge_+(E, \mathcal{K}) \bigoplus \bigwedge_-(E, \mathcal{K}) \tag{2.37}$$

with

$$\bigwedge_+(E, \mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^{2p}(E, \mathcal{K}), \quad \bigwedge_-(E, \mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^{2p+1}(E, \mathcal{K}), \tag{2.38}$$

which are the closed subspaces generated by vectors of even and odd degrees, respectively. Thus $\wedge(E,\mathcal{K})$ is $\mathbb{Z}_2$-graded. The grading operator of this graduation is given by

$$\Gamma = \mathrm{e}^{\mathrm{i}\pi I\otimes N_{\mathrm{f}}}. \tag{2.39}$$

Let $A$ and $B$ be strictly positive self-adjoint operators in $\mathcal{H}$ and $\mathcal{K}$, respectively, such that $A^{-\gamma}\in\mathcal{I}_1(\mathcal{H})$ for some $\gamma>0$ and $\exp(-tB)\in\mathcal{I}_1(\mathcal{K})$ for all $t>0$. Let $V$ be a measurable function on $E$ and $W$ be an $\mathcal{I}_2(\mathcal{K}_{\mathrm{c}})$-valued function on $E$ such that $W(\phi)$ is self-adjoint for a.e. $\phi$. Let

$$H = \mathrm{d}\Gamma_{\mathrm{b}}(A)\otimes I + V\otimes I + I\otimes \mathrm{d}\Gamma_{\mathrm{f}}(B) + \langle b^*|W|b\rangle. \tag{2.40}$$

By lemma 2.6, we have

$$[D(\mathrm{d}\Gamma_{\mathrm{b}}(A))\cap D(V)\cap D(\|W\|_2)]\,\widehat{\otimes}\,\wedge_{\mathrm{f}}(\mathcal{K}_{\mathrm{c}};B)\subset D(H). \tag{2.41}$$

Taking this fact into account, *we assume that there exists a dense subspace* $D\subset D(\mathrm{d}\Gamma_{\mathrm{b}}(A))\cap D(V)\cap D(\|W\|_2)$ *such that* $H$ *is bounded from below on* $D\,\widehat{\otimes}\,\wedge_{\mathrm{f}}(\mathcal{K}_{\mathrm{c}};B)$ and denote the Friedrichs extension of $H\restriction D\,\widehat{\otimes}\,\wedge_{\mathrm{f}}(\mathcal{K}_{\mathrm{c}};B)$ by the same symbol. We give a sufficient condition for $\mathrm{e}^{-tH}$ $(t>0)$ to be trace class on $\wedge(E,\mathcal{K})$ and a path integral representation for $\mathrm{Tr}(\Gamma^n\mathrm{e}^{-tH})$ $(n=0,1)$.

Let $\beta>0$. For each $\Phi\in E_\beta$, we define an operator $K_\pm(\Phi)$ on $L^2([0,\beta];\mathcal{K}_{\mathrm{c}})$ by

$$K_\pm(\Phi) = W(\Phi_t)\,((\partial_t)_\pm + B)^{-1}, \tag{2.42}$$

where $(\partial_t)_+$ $[(\partial_t)_-]$ denotes the differential operator $\partial/\partial t$ in $L^2([0,\beta])$ with the periodic [antiperiodic] boundary conditions.

**Lemma 2.8** [1,2,17]. *Suppose that, for a.e.* $\Phi$,

$$\int_0^\beta \|W(\Phi_t)\|_2^2\,\mathrm{d}t<\infty. \tag{2.43}$$

*Then* $K_\pm(\Phi)\in\mathcal{I}_2(L^2([0,\beta];\mathcal{K}_{\mathrm{c}}))$ *and, for each* $t,s\in[0,\beta]$, *there exist bounded linear operators* $K_\pm(\Phi;t,s)$ *on* $\mathcal{K}_{\mathrm{c}}$ *such that*

$$(K_\pm(\Phi)f)(t) = \int_0^\beta K_\pm(\Phi;t,s)f(s)\,\mathrm{d}s,\quad f\in L^2([0,\beta];\mathcal{K}_{\mathrm{c}}), \tag{2.44}$$

*and* $s\to K_\pm(\Phi;t,s)$ $(s\neq t)$ *is strongly continuous on* $\mathcal{K}_{\mathrm{c}}$. *Moreover,* $K_\pm(\Phi;t):=K_\pm(\Phi;t,t+0)\in\mathcal{I}_1(\mathcal{K}_{\mathrm{c}})$ *and*

$$\widetilde{\mathrm{Tr}}\,K_\pm(\Phi) = \int_0^\beta \mathrm{Tr}\;K_\pm(\Phi;t)\,\mathrm{d}t \tag{2.45}$$

*is finite and real.*

Let

$$H_0 = \mathrm{d}\Gamma_{\mathrm{b}}(A)\otimes I + I\otimes\mathrm{d}\Gamma_{\mathrm{f}}(B). \tag{2.46}$$

Then we have

$$Z(\beta;n) := \operatorname{Tr}\left(\Gamma^n \mathrm{e}^{-\beta H_0}\right) = \frac{\det(I + (-1)^n \mathrm{e}^{-tB})}{\det(I - \mathrm{e}^{-tA})}. \tag{2.47}$$

**Theorem 2.9**. *Let $A, B, V$ and $W$ be as above. Assume (2.43) and*

$$\int_{E_\beta} \mathrm{d}\mu_\beta(\Phi) \exp\left(-\int_0^\beta V(\Phi_t)\,\mathrm{d}t + \tfrac{1}{2}\|K_\pm(\Phi)\|_2^2 + |\widetilde{\operatorname{Tr}}\, K_\pm(\Phi)|\right) < \infty.$$

*Then* $\exp(-\beta H) \in \mathcal{I}_1(\wedge(E,\mathcal{K}))$ *and*

$$\frac{\operatorname{Tr}\left(\Gamma^{n(\sigma)}\mathrm{e}^{-\beta H}\right)}{Z(\beta;n(\sigma))} = \int_{E_\beta} \mathrm{d}\mu_\beta(\Phi)\ \det{}_2(I + K_\sigma(\Phi)) \times \exp\left(-\int_0^\beta V(\Phi_t)\,\mathrm{d}t + \widetilde{\operatorname{Tr}}\, K_\sigma(\Phi)\right), \tag{2.48}$$

*where* $n(+) = 1$ *and* $n(-) = 0$.

The proof of this theorem can be done by using theorem 2.5 and trace formulae in the Fermion Fock space $\wedge(\mathcal{K}_\mathrm{c})$ [1,2] together with some limiting arguments. The details are given in ref. [17].

## 3. Operators of the de Rham type, cohomology spaces and Laplacians

Let $S$ be a densely defined closed linear operator from $\mathcal{H}_\mathrm{c}$ to $\mathcal{K}_\mathrm{c}$. For each $p \geq 0$, we introduce a dense subspace $\mathfrak{D}_{S,p}$ in $\wedge^p(E,\mathcal{K})$ by

$$\mathfrak{D}_{S,p} = \mathcal{L}\{P(\phi(f_1),\ldots,\phi(f_n))u_1 \wedge \cdots \wedge u_p | P \in \mathbb{P}_n, f_j \in C^\infty(S^*S), u_k \in C^\infty(SS^*), j = 1,\ldots,n, k = 1,\cdots,p,n\}, \tag{3.1}$$

where $C^\infty(T) = \cap_{m=1}^\infty D(T^m)$. We define a linear operator $d_{S,p} : \wedge^p(E,\mathcal{K}) \to \wedge^{p+1}(E,\mathcal{K})$ with domain $\mathfrak{D}_{S,p}$ by

$$d_{S,p}P(\phi(f_1),\ldots,\phi(f_n))u_1 \wedge \cdots \wedge u_p = \sqrt{p+1}\sum_{j=1}^n (\partial_j P)(\phi(f_1),\ldots,\phi(f_n))Sf_j \wedge u_1 \wedge \cdots \wedge u_p \tag{3.2}$$

and extending it by linearity to $\mathfrak{D}_{S,p}$. The well-definedness of $d_{S,p}$ can be proven by using (2.8). Fundamental properties of the operator $d_{S,p}$ are summarized in the following proposition.

**Lemma 3.1** [1,2].

*(1) For all $p \geq 0$, $d_{S,p}$ is closable and its closure, also denoted by the same symbol, satisfies*

$$d_{S,p+1} d_{S,p} = 0 \quad \text{on } D(d_{S,p}). \tag{3.3}$$

*(2) For all $p \geq 0$, $D(d_{S,p}^*) \supset \mathfrak{D}_{S,p+1}$ and, for all $\Psi \in \mathfrak{D}_{S,p+1}$ of the form $\Psi = P(\phi(f_1), \ldots, \phi(f_n)) u_1 \wedge \cdots \wedge u_{p+1}$,*

$$d_{S,p}^* \Psi = \frac{1}{\sqrt{p+1}} \sum_{k=1}^{p+1} (-1)^{k-1} \left( \widetilde{P} \phi(S^* u_k) - \tilde{\nabla}_{J_{\mathcal{H}} S^* u_k} \widetilde{P} \right) \times u_1 \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge u_{p+1}, \tag{3.4}$$

*where $\widetilde{P} = P(\phi(f_1), \ldots, \phi(f_n))$ and $\hat{u}_k$ indicates the omission of $u_k$.*

Equation (3.3) shows that the sequence $\{d_{S,p}, D(d_{S,p})\}_{p=0}^{\infty}$ forms a complex of the de Rham type. We define the $p$-cohomology space of $\{d_{S,p}\}_{p=0}^{\infty}$ by

$$H_{S,p} = \mathrm{Ker}\ d_{S,p} / \overline{R(d_{S,p-1})}, \quad p \geq 0, \tag{3.5}$$

where $R(T)$ and Ker $T$ denote the range and the kernel of the operator $T$, respectively, and we set $R(d_{S,-1}) = \{0\} \subset \wedge^0(E, \mathcal{K})$. To identify these cohomology spaces, we introduce the Laplacians $\Delta_{S,p}$ defined by

$$\Delta_{S,p} = d_{S,p}^* d_{S,p} + d_{S,p-1} d_{S,p-1}^*, \tag{3.6}$$

which is a priori a nonnegative symmetric operator with $D(\Delta_{S,p}) \supset \mathfrak{D}_{S,p}$. We can prove the following theorem.

**Theorem 3.2** [1,2].

*(1) For all $p \geq 0$, $\Delta_{S,p}$ is self-adjoint and*

$$\Delta_{S,p} = \mathrm{d}\Gamma_{\mathrm{b}}(S^* S) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{f}}^{(p)}(S S^*). \tag{3.7}$$

*(2) For all $p \geq 0$, $H_{S,p}$ is isomorphic to* Ker $\Delta_{S,p}$.

We can identify Ker $\Delta_{S,p}$: Let $\Gamma_0(\mathrm{Ker}\ S) = \Gamma_0(\mathcal{H}) = \mathbb{C}$ and

$$\Gamma_r(\mathrm{Ker}\ S) = \overline{\mathcal{L}\{:\phi(f_1) \cdots \phi(f_r): \mid f_j \in \mathrm{Ker}\ S, j = 1, \cdots, r\}}, \quad r \geq 1. \tag{3.8}$$

Then we have

**Lemma 3.3** [2]. *For all $p \geq 0$,*

$$\mathrm{Ker}\ \Delta_{S,p} = \bigoplus_{r=0}^{\infty} \Gamma_r(\mathrm{Ker}\ S) \otimes A_p(\mathrm{Ker}\ S^* \otimes \cdots \otimes \mathrm{Ker}\ S^*). \tag{3.9}$$

We denote by $n(T)$ the dimension of Ker $T$:

$$n(T) = \dim \operatorname{Ker} T. \tag{3.10}$$

Using lemma 3.3, we obtain:

**Theorem 3.4.**
*(1) If $n(S) = 0$, then $n(\Delta_{S,0}) = 1$ and, for all $p \geq 1$,*

$$n(\Delta_{S,p}) = \begin{cases} \binom{n(S^*)}{p}, & 1 \leq p \leq n(S^*), \\ 0, & p > n(S^*). \end{cases}$$

*(2) If $n(S) \geq 1$, then $n(\Delta_{S,0}) = +\infty$ and, for all $p \geq 1$,*

$$n(\Delta_{S,p}) = \begin{cases} +\infty, & 1 \leq p \leq n(S^*), \\ 0, & p > n(S^*). \end{cases}$$

## 4. Free Dirac operators and Laplacians in the BFFS

The sequence $\{d_{S,p}\}_{p=0}^{\infty}$ of the de Rham operators defines an operator $d_S$ acting in $\wedge(E, \mathcal{K})$:

$$D(d_S) = \{\Psi = \{\Psi^{(p)}\}_{p=0}^{\infty} \in \wedge(E,\mathcal{K}) | \Psi^{(p)} \in D(d_{S,p}), \\ \textstyle\sum_{p=0}^{\infty} \|d_{S,p}\Psi^{(p)}\|^2 < \infty\}, \tag{4.1}$$

$$(d_S\Psi)^{(0)} = 0, \quad (d_S\Psi)^{(p)} = d_{S,p-1}\Psi^{(p-1)}, \quad p \geq 1. \tag{4.2}$$

**Lemma 4.1** [2].
*(1) The operator $d_S$ is densely defined, closed and satisfies*

$$d_S^2 = 0. \tag{4.3}$$

*(2) The adjoint $d_S^*$ is given by*

$$(d_S^*\Psi)^{(p)} = d_{S,p}^*\Psi^{(p+1)}, \quad p \geq 0, \tag{4.4}$$

*with domain*

$$D(d_S^*) = \{\Psi = \{\Psi^{(p)}\}_{p=0}^{\infty} \in \wedge(E,\mathcal{K}) | \Psi^{(p+1)} \in D(d_{S,p}^*), p \geq 0, \\ \textstyle\sum_{p=0}^{\infty} \|d_{S,p}^*\Psi^{(p+1)}\|^2 < \infty\}. \tag{4.5}$$

Equation (4.3) shows that $d_S$ is a nilpotent operator. Moreover, it follows from the closedness of $d_S$ that $\overline{R(d_S)} \subset$ Ker $d_S$. Hence we define the $d_S$-cohomology space by

$$H_S = \operatorname{Ker} d_S / \overline{R(d_S)}. \tag{4.6}$$

Let

$$\mathfrak{D}_S = \{\Psi = \{\Psi^{(p)}\}_{p=0}^{\infty} \in \wedge(E,\mathcal{K}) | \Psi^{(p)} \in \mathfrak{D}_{S,p}, \Psi^{(p)} = 0 \text{ for all but finitely many } p\}. \tag{4.7}$$

We define the Laplacian associated with $d_S$ by

$$\Delta_S = d_S^* d_S + d_S d_S^*, \tag{4.8}$$

which is a priori a nonnegative symmetric operator with $D(\Delta_S) \supset \mathfrak{D}_S$. Then we have:

**Theorem 4.2** [2].
*(1) The operator $\Delta_S$ is self-adjoint and*

$$\Delta_S = \bigoplus_{p=0}^{\infty} \Delta_{S,p} = \mathrm{d}\Gamma_{\mathrm{b}}(S^*S) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{f}}(SS^*). \tag{4.9}$$

*(2) The cohomology space $H_S$ is isomorphic to* Ker $\Delta_S$.

By (4.9) we have

$$\operatorname{Ker} \Delta_S = \bigoplus_{p=0}^{\infty} \operatorname{Ker} \Delta_{S,p}, \tag{4.10}$$

which implies that

$$\dim \operatorname{Ker} \Delta_S = \sum_{p=0}^{\infty} \dim \operatorname{Ker} \Delta_{S,p}.$$

Hence, by lemma 3.3 and theorem 3.4, we obtain

$$\operatorname{Ker} \Delta_S = \bigoplus_{p,r=0}^{\infty} \Gamma_r(\operatorname{Ker} S) \otimes A_p(\operatorname{Ker} S^* \otimes \cdots \otimes \operatorname{Ker} S^*) \tag{4.11}$$

$$\dim \operatorname{Ker} \Delta_S = \begin{cases} 2^{n(S^*)}, & \text{if } n(S) = 0, \\ +\infty, & \text{if } n(S) \geq 1. \end{cases} \tag{4.12}$$

We define

$$Q_S = d_S + d_S^*. \tag{4.13}$$

Then we have:

**Theorem 4.3** [1,2].
*(1) The operator $Q_S$ is self-adjoint, essentially self-adjoint on every core of $\Delta_S$, and the following operator equalities hold:*

$$\Delta_S = Q_S^2 = Q_{\mathrm{i}S}^2. \tag{4.14}$$

*(2) The grading operator $\Gamma$ leaves $D(Q_S)$ invariant and*

$$\{Q_S, \Gamma\} = 0 \quad on\ D(Q_S). \tag{4.15}$$

Theorem 4.3(2) shows that $Q_S$ is an abstract Dirac operator (see section 5). Note that

$$Q_{iS} = i(d_S - d_S^*), \tag{4.16}$$

i.e., $Q_{iS}$ is an operator of the Kähler–Dirac type. Taking these facts and (4.14) into account, we may call $Q_S$ a *"free" (Kähler-)Dirac operator in the BFFS* $\wedge(E, \mathcal{K})$.

We can easily check that

$$\{Q_S, Q_{iS}\} = 0 \quad \text{on } D(d_S^* d_S) \cap D(d_S d_S^*), \tag{4.17}$$

i.e., $Q_S$ and $Q_{iS}$ anticommute in a "naive" sense. In fact, we can prove a stronger result on the anticommutativity of them in the sense made precise below.

Let us recall a proper notion of anticommutativity of two self-adjoint operators [33,39]: Two self-adjoint operators $A$ and $B$ in a Hilbert space are said to *strongly anticommute* if, for all $t \in \mathbb{R}$ and $f \in D(B)$, $e^{itA} f \in D(B)$ and $e^{itA} B f = B e^{-itA} f$. We remark that, if $A$ and $B$ are strongly anticommuting self-adjoint operators, then there exists a dense invariant subspace $\mathcal{D}$ for $A$ and $B$ such that $AB + BA = 0$ on $\mathcal{D}$. But the converse is not true in general. Strongly anticommuting self-adjoint operators have interesting properties [13].

We can prove the following fact.

**Theorem 4.4** [5]. *The Dirac operators $Q_S$ and $Q_{iS}$ strongly anticommute.*

## 5. Elements of index theory and supersymmetry

Our infinite dimensional analysis has a connection with index theory and supersymmetry. To make this aspect clear, here we take an interlude to review some basic facts in index theory and in supersymmetric quantum theory.

### 5.1. FREDHOLM OPERATORS AND INDEX

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of densely defined closed linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. Then the index of $A$ is defined by

$$\text{ind } A = n(A) - n(A^*), \tag{5.1}$$

provided that at least one of $n(A)$ and $n(A^*)$ is finite. An operator $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be Fredholm (semi-Fredholm) if both (at least one) of $n(A)$ and $n(A^*)$ are (is) finite and the range $R(A)$ of $A$ is closed. It can be shown that $A$ is (semi-)

Fredholm if and only if so is $A^*$. An important feature of a (semi-) Fredholm operator is the "topological invariance" of the index : *Let* $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ *be (semi-) Fredholm and B be an A-compact operator. Then* $A + B$ *is (semi-) Fredholm and* ind $(A + B) =$ ind $A$ (e.g, [32, ch. IV, §5], [22, §XVII.4]).

A useful criterion for an operator $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ to be (semi-) Fredholm is given in terms of spectral properties of the nonnegative self-adjoint operators $A^*A$ and $AA^*$:

**Proposition 5.1.** *Let* $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. *Then the following hold:*

*(1) The operator A is Fredholm if and only if* $n(A^*A) < \infty$, $n(AA^*) < \infty$, *and*

$$\inf \sigma(A^*A) \setminus \{0\} > 0, \tag{5.2}$$

*where* $\sigma(\cdot)$ *denotes spectrum.*

*(2) The operator A is semi-Fredholm if and only if at least one of* $n(A^*A)$ *and* $n(AA^*)$ *is finite and (5.2) holds.*

*Proof (outline).* We need only to employ the following general facts :

(i)

$$\text{Ker } A^*A = \text{Ker } A. \tag{5.3}$$

(ii) $\sigma(A^*A) \setminus \{0\} = \sigma(AA^*) \setminus \{0\}$ (see ref. [20]).

(iii) $R(A)$ is closed if and only if there exists a constant $C > 0$ such that $||Af|| \geq C||f||$ for all $f \in D(A) \cap (\text{Ker } A)^{\perp}$. □

### 5.2. SUPERSYMMETRIC QUANTUM THEORY AND ABSTRACT DIRAC OPERATORS

As we have already seen, the BFFS $\wedge(E, \mathcal{K})$ is $\mathbb{Z}_2$-graded and the free Dirac operator $Q_S$ is odd w.r.t. the grading in the sense that $Q_S$ anticommutes with the grading operator $\Gamma$. This is the reflection of a supersymmetric structure of the BFFS. In order to make this aspect clear, we briefly discuss in this subsection supersymmetric quantum theory (SSQT) in an abstract way [8,9,14,19,40].

We begin with an abstract definition of SSQT. For simplicity, we consider only the case of $N = 1$ supersymmetry .

**Definition 5.2.** An $(N = 1)$ SSQT is a quadruple $\{\mathcal{X}, H, Q, N_F\}$ consisting of a Hilbert space $\mathcal{X}$, self-adjoint operators $H$ (the " supersymmetric Hamiltonian"), $Q$ (the "supercharge") and $N_F$ (the "Fermion number operator") in $\mathcal{X}$ with the following properties:

(S.1) $N_F$ is bounded with $N_F^2 = I$ and $N_F \neq \pm I$.

(S.2) $N_F$ leaves $D(Q)$ invariant and $\{Q, N_F\} = 0$ on $D(Q)$.

(S.3) $H = Q^2$.

The results in section 4 show that $\{\Lambda(E,\mathcal{K}), \Delta_S, Q_S, \Gamma\}$ is a SSQT. In section 9 we discuss concrete realizations of this SSQT as SSQFT models.

Let $\{\mathcal{X}, H, Q, N_F\}$ be a SSQT. Then, by (S.1), we have $\sigma(N_F) = \{\pm 1\}$. Hence we can decompose $\mathcal{X}$ as

$$\mathcal{X} = \mathcal{X}_+ \bigoplus \mathcal{X}_- \tag{5.4}$$

with $\mathcal{X}_\pm$ being the eigenspace of $N_F$ with eigenvalue $\pm 1$. Physically, $\mathcal{X}_+$ and $\mathcal{X}_-$ correspond to the space of bosonic and fermionic states, respectively. The decomposition (5.4) allows one to represent vectors $f$ in $\mathcal{X}$ as

$$f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

with $f_\pm \in \mathcal{X}_\pm$. Then, relative to this representation, every linear operator in $\mathcal{X}$ can be represented as a $2 \times 2$ matrix with entries being linear operators. It follows from the self-adjointness of $Q$ that there exists a unique $Q_+ \in C(\mathcal{X}_+, \mathcal{X}_-)$ such that

$$Q = \begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix}. \tag{5.5}$$

Hence $H$ is represented as

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{5.6}$$

with

$$H_+ = Q_+^* Q_+, \quad H_- = Q_+ Q_+^*, \tag{5.7}$$

which are called the bosonic and fermionic part of $H$, respectively.

The supersymmetry is said to be broken if there exist no zero-energy states, in other words, Ker $H = \{0\}$ [40,41]. A necessary condition for the supersymmetry to be broken is given in terms of the *Witten index* defined by

$$I_W = n(H_+) - n(H_-), \tag{5.8}$$

which physically means the number of the bosonic zero-energy states minus the number of the fermionic zero-energy states. It is obvious that, if the supersymmetry is broken, then $I_W = 0$.

By (5.3) and (5.7), we see that

$$I_W = \text{ind } Q_+. \tag{5.9}$$

Thus the index of $Q_+$ is related to the physics of supersymmetry. In view of the topological invariance of the index of (semi-) Fredholm operator, it is important to know when $Q_+$ is (semi-) Fredholm.

**Lemma 5.3.** *The operator $Q_+$ is Fredholm if and only if $Q$ is Fredholm.*

*Proof.* Apply the fact (iii) given in the proof of proposition 5.1. □

**Lemma 5.4.** *The operator $Q_+$ is Fredholm if and only if*

$$n(Q^2) < \infty, \quad \inf\, \sigma(Q^2) \setminus \{0\} > 0. \tag{5.10}$$

*Proof.* This follows from lemma 5.3 and proposition 5.1 (1) applied to $A = Q$. □

The following lemma is also useful.

**Lemma 5.5.** *Suppose that for some $\beta > 0$, $\exp(-\beta Q^2) \in \mathcal{I}_1(\mathcal{X})$. Then $Q_+$ is Fredholm and*

$$\mathrm{ind}\ Q_+ = \mathrm{Tr}\left(N_{\mathrm{F}} \mathrm{e}^{-\beta Q^2}\right) \tag{5.11}$$

*independently of $\beta$.*

*Proof.* See ref. [1, appendix B]. □

We conclude this section with a mathematical remark. A closed symmetric operator $\mathbb{D}$ (not necessarily self-adjoint) in the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{X}$ is said to be an *abstract Dirac operator w.r.t. the grading operator* $N_{\mathrm{F}}$ if $N_{\mathrm{F}}$ leaves $D(\mathbb{D})$ invariant and $\{\mathbb{D}, N_{\mathrm{F}}\} = 0$ on $D(\mathbb{D})$ [31]. In terms of this notion, the supercharge $Q$ is an abstract Dirac operator. Hence the abstract SSQT can be regarded as a theory of an abstract Dirac operator, which has rather rich structures (e.g., [8,9,13,14,19,29] and references therein).

## 6. Fredholm property of the free Dirac operators

We now turn to the free Dirac operators introduced in section 4 and examine their Fredholm property. We already have seen that $\{\wedge(E,\mathcal{K}), \Delta_S, Q_S, \Gamma\}$ is a SSQT. Hence there exists a unique $Q_{S,+} \in \mathcal{C}(\wedge_+(E,\mathcal{K}), \wedge_-(E,\mathcal{K}))$ such that

$$Q_S = \begin{pmatrix} 0 & Q_{S,+}^* \\ Q_{S,+} & 0 \end{pmatrix}. \tag{6.1}$$

**Theorem 6.1** [2].
*(1) If $S$ is Fredholm with $n(S) = 0$, then $Q_{S,+}$ is Fredholm with*

$$\mathrm{ind}\ Q_{S,+} = \delta_{0,-\mathrm{ind}\ S} \tag{6.2}$$

*(2) If $S$ is semi-Fredholm with $n(S) \geq 1$ and $n(S^*) = 0$, then $Q_{S,+}$ is semi-Fredholm with*

$$\mathrm{ind}\ Q_{S,+} = n(Q_{S,+}) = +\infty. \tag{6.3}$$

*Proof (outline).* Apply lemma 5.4 with $Q = Q_S$ together with (4.14), (4.9), and theorem 3.4. □

**Remark 6.2.** In the cases different from those considered in theorem 6.1, $Q_{S,+}$ is not (semi-) Fredholm.

## 7. Perturbation of the free Dirac operators

In this section we consider a perturbation of $Q_S$. This is done by perturbing the de Rham operator $d_S$.

For $F \in L^2(E, \mathrm{d}\mu; \mathcal{K}_c)$, we define an operator $\widetilde{b}(F)$ in $\wedge(E, \mathcal{K})$ by

$$(\widetilde{b}(F)\Psi)(\phi) = b(F(\phi))\Psi(\phi), \quad \text{a.e. } \phi \in E, \tag{7.1}$$

with

$$D(\widetilde{b}(F)) = \{\Psi \in \wedge(E, \mathcal{K}) | \int_E \|b(F(\phi))\Psi(\phi)\|^2_{\wedge(\mathcal{K}_c)} \mathrm{d}\mu(\phi) < \infty\}. \tag{7.2}$$

We introduce a perturbed de Rham operator by

$$d_S(F) = d_S + \widetilde{b}(F)^*. \tag{7.3}$$

**Lemma 7.1** [2]. *Let $F \in L^q(E, \mathrm{d}\mu; \mathcal{K}_c)$ with some $q > 2$. Then, $d_S(F)$ is closable with $D(d_S(F)^\#) \supset \mathfrak{D}_S$, where $\mathfrak{D}_S$ is defined by (4.7).*

**Remark 7.2.** By Hölder's inequality, $F \in L^2(E, \mathrm{d}\mu; \mathcal{K}_c)$ if $F \in L^q(E, \mathrm{d}\mu; \mathcal{K}_c)$ with some $q > 2$.

For $F \in L^q(E, \mathrm{d}\mu; \mathcal{K}_c)$ with some $q > 2$, we define a perturbed Dirac operator $Q_S(F)$ by

$$Q_S(F) = \text{the closure of } (d_S(F) + d_S(F)^*) \restriction \mathfrak{D}_S, \tag{7.4}$$

which is a closed symmetric operator. Then we can prove:

**Lemma 7.3** [1,2]. *The operator $Q_S(F)$ is an abstract Dirac operator w.r.t. $\Gamma$.*

In general, it is shown that an abstract Dirac operator always has a self-adjoint extension which is also an abstract Dirac operator ([31], cf. also [12,15]). Hence, by lemma 7.3, $Q_S(F)$ has a self-adjoint extension as an abstract Dirac operator. It is interesting and important to discuss the essential self-adjointness of $Q_S(F)$. In the present paper, however, we do not consider this aspect and simply assume the following:

**Assumption I.** The operator $Q_S(F)$ is essentially self-adjoint on $\mathfrak{D}_S$.

The problem of the essential self-adjointness of $Q_S(F)$ will be discussed in a separate paper [16].

By lemma 7.3 and the self-adjointness of $Q_S(F)$, there exists a unique $Q_S(F)_+ \in \mathcal{C}(\wedge_+(E,\mathcal{K}), \wedge_-(E,\mathcal{K}))$ such that

$$Q_S(F) = \begin{pmatrix} 0 & Q_S(F)_+^* \\ Q_S(F)_+ & 0 \end{pmatrix}. \tag{7.5}$$

We next consider the Fredholm property of $Q_S(F)_+$ and compute its index by the method of lemma 5.5. To do that, however, we need to know an explicit form of the *Laplacian*

$$\Delta_S(F) = Q_S(F)^2. \tag{7.6}$$

As in lemma 2.1, we can show that the operator $S\nabla : L^2(E, \mathrm{d}\mu) \to L^2(E, \mathrm{d}\mu; \mathcal{K}_c)$ with domain $\mathcal{P}_S$ is closable. We denote the closure by the same symbol. In the subspace $\mathcal{P}_S \hat{\otimes} D(S^*)$ in $L^2(E, \mathrm{d}\mu; \mathcal{K}_c) = L^2(E, \mathrm{d}\mu) \otimes \mathcal{K}_c$, we define a norm $\|\cdot\|_{r,s}$ $[r, s \in [1, \infty)]$ by

$$\begin{aligned} \|\Psi\|_{r,s} = \|\Psi\|_{L^r(E,\mathrm{d}\mu;\mathcal{K}_c)} &+ \|S\nabla \otimes I\Psi\|_{L^s(E,\mathrm{d}\mu;\mathcal{K}_c)\otimes\mathcal{K}_c} \\ &+ \|SJ_{\mathcal{H}}\nabla \otimes J_{\mathcal{K}}\Psi\|_{L^s(E,\mathrm{d}\mu;\mathcal{K}_c)\otimes\mathcal{K}_c} \end{aligned} \tag{7.7}$$

and denote by $W_S^{r,s}$ the completion of $\mathcal{P}_S \hat{\otimes} D(S^*)$ in the norm $\|\cdot\|_{r,s}$.

**Definition 7.4.** We say that a $\mathcal{K}_c$-valued function $F$ on $E$ is in the set $\mathbb{F}_S^{r,s}$ if $F \in D(\nabla^* S^*) \cap W_S^{r,s}$ and $J_{\mathcal{H}} S^* F = S^* F$.

Let $G$ be a $\mathcal{K}_c \otimes \mathcal{K}_c$-valued measurable function on $E$. Then, for each $\phi \in E$, $G(\phi)$ is an element of $\mathcal{K}_c \otimes \mathcal{K}_c$. Hence, by lemma 2.7, we can define three quadratic operators $\langle b^\# | C_{G(\phi)} | b^\# \rangle$ in $\wedge(\mathcal{K}_c)$. These operators naturally define operators acting in $\wedge(E, \mathcal{K})$ by

$$(\langle b^\# | C_G | b^\# \rangle \Psi)(\phi) = \langle b^\# | C_{G(\phi)} | b^\# \rangle \Psi(\phi) \tag{7.8}$$

with the domain being maximal.

**Theorem 7.5** [2]. *Let $F \in \mathbb{F}_S^{r,s}$ with $r > 4$ and $s > 2$. Let*

$$L_{\mathrm{F}}(\phi) = C_{(SJ_{\mathcal{H}}\nabla\otimes J_{\mathcal{K}}F)(\phi)} + C^*_{(SJ_{\mathcal{H}}\nabla\otimes J_{\mathcal{K}}F)(\phi)}. \tag{7.9}$$

*Then $\mathfrak{D}_S \subset D(\Delta_S(F))$ and*

$$\begin{aligned} \Delta_S(F) = \Delta_S &+ \nabla^* S^* F + \|F\|^2_{\mathcal{K}_c} \\ &+ \langle b^* | C_{S\nabla\otimes IF} | b^* \rangle + \langle b | C^*_{S\nabla\otimes IF} | b \rangle + \langle b^* | L_{\mathrm{F}} | b \rangle \end{aligned} \tag{7.10}$$

*on $\mathfrak{D}_S$.*

## 8. Path integral representation of the index of the perturbed Dirac operator

In this section we state a result on a path integral representation of the index of $Q_S(F)_+$. We assume the following:

**Assumption II.** $S^*S$ and $SS^*$ are strictly positive and $(S^*S)^{-\gamma} \in \mathcal{I}_1(\mathcal{H})$ for some $\gamma > 0$.

Under assumption II, as was discussed in section 2, we can take

$$E = \mathcal{H}_{-\delta}(S^*S) \tag{8.1}$$

with $\delta > \gamma$. We denote by $\{\Phi_t | t \in [0,\beta]\}$ the Gaussian random process given in proposition 2.2 with $A = S^*S$ and by $(E_\beta, \mu_\beta)$ the underlying measure space. Let $F \in \mathbb{F}_S^{r,s}$ with $r > 4, s > 2$. For a.e. $\Phi \in E_\beta$, we define an operator on $L^2([0,\beta]; \mathcal{K}_c)$ by

$$K_\pm^F(\Phi) = L_F(\Phi_t)((\partial_t)_\pm + SS^*)^{-1}. \tag{8.2}$$

The following lemma follows from an application of lemma 2.8.

**Lemma 8.1.** *Suppose that*

$$\int_0^\beta ||L_F(\Phi_t)||_2^2 \, dt < \infty. \tag{8.3}$$

*Then* $K_\pm^F(\Phi) \in \mathcal{I}_2(L^2([0,\beta];\mathcal{K}_c))$ *and there exist bounded linear operators* $K_\pm^F(\Phi; t, s) : \mathcal{K}_c \to \mathcal{K}_c$ *such that* $s \to K_\pm^F(\Phi; t, s)$ $(s \neq t)$ *is strongly continuous and*

$$(K_\pm^F(\Phi) f)(t) = \int_0^\beta K_\pm^F(\Phi; t, s) f(s) \, ds, \quad f \in L^2([0,\beta]; \mathcal{K}_c).$$

*Moreover,* $K_\pm^F(\Phi; t) := K_\pm^F(\Phi; t, t+0) \in \mathcal{I}_1(\mathcal{K}_c)$ *and*

$$\widetilde{\mathrm{Tr}} \; K_\pm^F(\Phi) = \int_0^\beta \mathrm{Tr} \, K_\pm^F(\Phi; t) \, dt$$

*is finite and real.*

A formula for the index of $Q_S(F)_+$ is given in the following theorem.

**Theorem 8.2** [2]. *Consider the case*

$$C_{S\nabla \otimes IF}^* = J_\mathcal{K} C_{S\nabla \otimes IF} J_\mathcal{K}. \tag{8.4}$$

*Assume (8.3) and*

$$\int_{E_\beta} \mathrm{d}\mu_\beta(\Phi) \exp\left( - \int_0^\beta \left( \nabla^* S^* F(\Phi_t) + \|F(\Phi_t)\|^2_{\mathcal{K}_c} \right) \mathrm{d}t \right.$$
$$\left. + \tfrac{1}{2} \|K^F_\pm(\Phi)\|_2^2 + |\widetilde{\mathrm{Tr}}\, K^F_\pm(\Phi)| \right) < \infty .$$

*Then $Q_S(F)_+$ is Fredholm and*

$$\operatorname{ind} Q_S(F)_+ = \int_{E_\beta} \mathrm{d}\mu_\beta(\Phi) \det{}_2 \left(I + K^F_+(\Phi)\right)$$
$$\times \exp\left( - \int_0^\beta \left( \nabla^* S^* F(\Phi_t) + \|F(\Phi_t)\|^2_{\mathcal{K}_c} \right) \mathrm{d}t + \widetilde{\mathrm{Tr}}\; K^F_+(\Phi) \right),$$

*independently of $\beta > 0$.*

This theorem can be proven by applying theorem 2.9 and lemma 5.5.

**Remark 8.3.** It is shown that (8.4) is equivalent to the condition

$$S\nabla \otimes IF(\phi) \in \Lambda^2(\mathcal{K}_c)^\perp \quad \text{a.e. } \phi .$$

In this case we have $\langle b^* | C_{S\nabla\otimes IF} | b^* \rangle = 0$, which simplifies the computation of the index of $Q_S(F)_+$. The case where (8.4) is not satisfied is more complicated, see ref. [2].

## 9. Models of SSQFT

The theory of infinite dimensional Dirac operators described in the preceding sections is constructed from the "sextet" $(E, \mu, \mathcal{H}, \mathcal{K}, S, F)$. In this section we briefly discuss concrete examples of it. In particular, we want to point out that the abstract formalism presented in this paper gives a mathematical unification for some models in SSQFT.

### 9.1. DIRAC OPERATORS ON THE SPACE OF REAL TEMPERED DISTRIBUTIONS AND THE $N = 1$ WESS–ZUMINO MODEL

Let $\mathcal{S}_r(\mathbb{R})$ be the Schwartz space of rapidly decreasing real $C^\infty$-functions on $\mathbb{R}$ and $\mathcal{S}_r(\mathbb{R})^*$ be its dual, i.e., the space of real tempered distributions on $\mathbb{R}$. Let $m > 0$ be a constant and

$$\omega(p) = \sqrt{p^2 + m^2}, \quad p \in \mathbb{R}.$$

Let $H_{-1/2}(\mathbb{R})$ be the real Hilbert space of real tempered distributions $f$ on $\mathbb{R}$ such that the Fourier transform $\hat{f}$ is a measurable function and

$$\|f\|^2_{-1/2} = \frac{1}{2} \int_{\mathbb{R}} \frac{|\hat{f}(p)|^2}{\omega(p)} < \infty .$$

We want to apply the abstract theory in the previous sections to the case where

$$E = \mathcal{S}_r(\mathbb{R})^*, \quad \mathcal{H} = H_{-1/2}(\mathbb{R}), \quad \mathcal{K} = L_r^2(\mathbb{R}),$$

and the measure $\mu$ is equal to the measure $\mu_m$ on $\mathcal{S}_r(\mathbb{R})^*$ such that

$$\int_{\mathcal{S}_r(\mathbb{R})^*} \exp(\mathrm{i}\langle\phi, f\rangle)\,\mathrm{d}\mu_m(\phi) = \exp(-\|f\|_{-1/2}^2/2), \quad f \in \mathcal{S}_r(\mathbb{R}),$$

where $\langle\phi, f\rangle$ denotes the canonical duality pairing between $\phi \in \mathcal{S}_r(\mathbb{R})^*$ and $f \in \mathcal{S}_r(\mathbb{R})$.

There is some arbitrariness in choosing the operator $S : H_{-1/2}(\mathbb{R})_{\mathrm{c}} \to L^2(\mathbb{R})$ to define the Dirac operator $Q_S$. One choice is given by the operator $S$ such that $S = S_1 + \mathrm{i}S_2$ with

$$(\widehat{S_1 f})(p) = \frac{\nu(p)}{2\sqrt{\omega(p)}}\hat{f}(p), \quad (\widehat{S_2 f})(p) = \frac{\nu(-p)}{2\sqrt{\omega(p)}}\hat{f}(p),$$

$$\nu(p) = \sqrt{p + \omega(p)}.$$

Let $\omega_{\mathrm{b}}$ and $\omega_{\mathrm{f}}$ be the self-adjoint operators acting in $H_{-1/2}(\mathbb{R})$ and $L_r(\mathbb{R})$, respectively, such that

$$\widehat{\omega_{\mathrm{b}} f}(p) = \omega(p)\hat{f}(p), \; f \in D(\omega_{\mathrm{b}}),$$
$$\widehat{\omega_{\mathrm{f}} u}(p) = \omega(p)\hat{u}(p), \; u \in D(\omega_{\mathrm{f}}).$$

Then we can easily show that

$$S^*S = \omega_{\mathrm{b}}, \quad SS^* = \omega_{\mathrm{f}}.$$

Thus, in the present case, the free Laplacian takes the form

$$\Delta_S = \mathrm{d}\Gamma_{\mathrm{b}}(\omega_{\mathrm{b}}) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{f}}(\omega_{\mathrm{f}}). \tag{9.1}$$

It may be instructive to write down an explicit form of the Dirac operator in the case under consideration. We denote by $b(x)$ and $b(x)^*$ the distribution kernels of the annihilation and creation operators on the Fermion Fock space $\wedge(L^2(\mathbb{R}))$, respectively. Let

$$\mathfrak{F} = \mathcal{L}\{P(\langle\phi, f_1\rangle, \ldots, \langle\phi, f_n\rangle)u_1 \wedge \cdots \wedge u_p | f_j, u_k \in \mathcal{S}(\mathbb{R}), j = 1, \ldots, n,$$
$$k = 1, \ldots, p, P \in \mathbb{P}_n, n \geq 0, p \geq 0\},$$

which is dense in $\wedge(\mathcal{S}_r(\mathbb{R})^*, L^2(\mathbb{R}))$. One can easily see that $D(d_S) \supset \mathfrak{F}$ and $d_S$ can be expressed as

$$d_S = \int \mathrm{d}x\, b(x)^* S\, \delta/\delta\phi(x) \quad \text{on } \mathfrak{F}, \tag{9.2}$$

where $\delta/\delta\phi(x)$ denotes the functional differential operator of first order. Let

$$\gamma_1(x) = \mathrm{i}[b(x) - b(x)^*], \quad \gamma_2(x) = b(x) + b(x)^*.$$

Then one has

$$\{\gamma_j(x), \gamma_k(y)\} = 2\delta_{jk}\delta(x - y), \tag{9.3}$$

i.e., $\{\gamma_j(x)\}$ is a set of (distributional) generators of an infinite dimensional Clifford algebra. We can easily check that the free Dirac operator $Q_S$ takes the form

$$Q_S = \mathrm{i}\sum_{j=1}^{2}\int \mathrm{d}x\,\gamma_j(x)S_j\frac{\delta}{\delta\phi(x)} + \int \mathrm{d}x\,\phi(x)S^*b(x) \quad \text{on } \mathfrak{F}, \tag{9.4}$$

which shows that $Q_S$ is a functional differential operator of first order with coefficients being elements of an infinite dimensional Clifford algebra. In this sense, $Q_S$ is certainly an infinite dimensional generalization of usual finite dimensional Dirac operators.

Using (9.1) or (9.4), we see that the Laplacian $\Delta_S = Q_S^2$ can be expressed as

$$\Delta_S = \int \mathrm{d}x\left(-\frac{\delta}{\delta\phi(x)}\omega_{\mathrm{b}}\frac{\delta}{\delta\phi(x)} + \phi(x)\omega_{\mathrm{b}}\frac{\delta}{\delta\phi(x)}\right) + \int \mathrm{d}x\,b(x)^*\omega_{\mathrm{f}}b(x) \quad \text{on } \mathfrak{F}. \tag{9.5}$$

We next give an example of $L^2(\mathbb{R})$-valued functions $F$ on $\mathcal{S}_r(\mathbb{R})^*$ used to define a perturbation of $Q_S$. Let $\kappa > 0$ and $\varrho$ be a nonnegative even function in $\mathrm{C}_0^\infty(\mathbb{R})$ such that $\varrho(x) = 0$ for $|x| \geq 1$ and $\int \varrho(x) = 1$. Then the field $\phi_\kappa(x)$ with the ultraviolet cutoff $\kappa$ is defined by $\phi_\kappa(x) = \langle\phi, \kappa\varrho(\kappa(x-\cdot))\rangle$. For $g \in \mathrm{C}_0^\infty(\mathbb{R}^2)$ and a polynomial $P$ of one variable, we can define an $L^2(\mathbb{R})$-valued function $F$ on $\mathcal{S}_r(\mathbb{R})^*$ by

$$F(\phi)(x) = \int_{\mathbb{R}} :P(\phi_\kappa(y)):g(x,y)\,\mathrm{d}y.$$

It is not difficult to check that $F \in \mathbb{F}_S^{r,s}$ with $r > 4, s > 2$. Thus, by theorem 7.5, we can explicitly write down $\Delta_S(F)$. But, here, we do not go into the details (cf. ref. [2]). The above choice of $F$ gives a SSQFT model of the $N = 1$ Wess–Zumino type in the two-dimensional space–time, which describes a supersymmetric system of two component Majorana Fermi fields interacting with a neutral scalar field. The case where $\mathbb{R}$ is replaced by the one-torus $\mathrm{S}^1$ is discussed in refs. [26,29,30].

### 9.2. THE $N = 2$ WESS–ZUMINO MODEL

A complex version of the model in the last subsection is described in terms of $(E, \mu, \mathcal{H}, \mathcal{K})$ given by

$$E = \mathcal{S}_r(\mathbb{R})^* \times \mathcal{S}_r(\mathbb{R})^*, \qquad \mu = \mu_m \otimes \mu_m,$$
$$\mathcal{H} = H_{-1/2}(\mathbb{R}) \bigoplus H_{-1/2}(\mathbb{R}), \qquad \mathcal{K} = L_r^2(\mathbb{R}) \bigoplus L_r^2(\mathbb{R}).$$

Let

$$\sigma(p) = -\mathrm{i}\,2^{-3/2}\omega(p)^{-1}[\nu(-p)^3 + m\nu(p)].$$

We define an operator $S : H_{-1/2}(\mathbb{R})_c \bigoplus H_{-1/2}(\mathbb{R})_c \to L^2(\mathbb{R}) \bigoplus L^2(\mathbb{R})$ by

$$(Sf)^\wedge(p) = \frac{1}{\sqrt{2}}\omega(p)^{-1/2}\begin{pmatrix}\sigma(-p)^* & \sigma(p)\\ \sigma(p) & -\sigma(-p)^*\end{pmatrix}\begin{pmatrix}\hat{f}_+(p)\\ \hat{f}_-(p)\end{pmatrix},$$

$$f = \begin{pmatrix}f_+\\ f_-\end{pmatrix} \in H_{-1/2}(\mathbb{R})_c \bigoplus H_{-1/2}(\mathbb{R})_c.$$

Then we can snow that this choice gives the $N = 2$ Wess–Zumino model on $\mathbb{R}^2$, which describes a supersymmetric quantum system of Fermi fields coupled to a complex Bose field. In our formalism, the complex Bose field is given by the random variable

$$\phi(f) = \frac{1}{\sqrt{2}}(\langle\phi_1, f\rangle + \mathrm{i}\langle\phi_2, f\rangle), \quad f \in \mathcal{S}_r(\mathbb{R}),$$

where $(\phi_1, \phi_2) \in E = \mathcal{S}_r(\mathbb{R})^* \times \mathcal{S}_r(\mathbb{R})^*$. The model on a cylinder was discussed in detail in refs. [26–29]. Mathematical discussions of the supersymmetric quantum mechanical version of this model have been given in refs. [10–12,25].

### 9.3. A MODEL OF SUPERSYMMETRIC GAUGE FIELD THEORY

In ref. [42], Witten presented a SSQFT model derived from the Chern–Simons functional on a three-manifold $Y$ and suggested that its ground states are the Floer groups of $Y$ (a conjecture given by Atiyah [18]). This model (with cutoffs) can also be described as a special example of our abstract formalism. It is interesting to remove the cutoffs and mathematically justify Atiyah–Witten's suggestion just mentioned.

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